CentER

Discussion Paper

No. 2010–116

THE MAXIMUM ORDER OF ADJACENCY MATRICES WITH A GIVEN RANK

By W.H. Haemers and M.J.P. Peeters

October 2010

ISSN 0924-7815

TILBURG UNIVERSITY

The maximum order of adjacency matrices with a given rank

W.H. Haemers* and M.J.P. Peeters

Department of Econometrics and Operations Research,

Tilburg University, The Netherlands

Dedicated to the 65th birthday of Richard M. Wilson

Abstract

We look for the maximum order $m(r)$ of the adjacency matrix A of a graph G with a fixed rank r, provided A has no repeated rows or all-zero row. Akbari, Cameron and Khosrovshahi conjecture that $m(r) = 2^{(r+2)/2} - 2$ if r is even, and $m(r) = 5 \cdot 2^{(r-3)/2} - 2$ if r is odd. We prove the conjecture and characterize G in the case that G contains an induced subgraph $\frac{r}{2}K_2$, or $\frac{r-3}{2}K_2 + K_3$.

Keywords: Graph, Adjacency matrix.
AMS Subject Classification: 05B20, 05C50. **JEL-code:** C0.

1 Introduction

In this paper we discus the problem of determining the maximum number of vertices of a graph in terms of the rank of its adjacency matrix (also called the rank of the graph). This problem is only properly defined if the matrix has no repeated rows. Given a graph we can duplicate a vertex arbitrarily often and add isolated vertices as many as we like without changing the rank. This motivates the following definition ([2]).

Definition 1 *A graph is* reduced *if it has no isolated vertices and no two vertices have the same set of neighbors.*

Although we find this definition more convenient, some authors ([3], [1]) use 'reduced' purely for the condition that no two vertices have the same set of neighbors and allow the graph to have at most one isolated vertex. This explains why there is sometimes a difference of one in their results and ours.

Let $m(r)$ be the maximum number of vertices in a reduced graph with rank r over $\mathbb{R}$. There are clearly no graphs of rank 1, and the only reduced graphs of rank 2 and 3 are K_2 and K_3 respectively. The number $m(r)$ can be defined for any field and for all these cases we have a trivial upper bound of $2^r - 1$. Godsil and Royle [2] prove that for the field $\mathbb{F}_2$, $m(r)$ is equal to this upper bound. In this paper we only consider $m(r)$ for the field of real numbers.

Kotlov and Lovász [3] improve the upper bound on $m(r)$ to $\mathcal{O}(2^{r/2})$. They also give a construction that transforms a reduced graph of rank r on n vertices into a reduced graph of rank $r+2$ on $2n+2$ vertices. This means that $m(r+2) \geq 2m(r)+2$, implying a lower bound of the same order as the upper bound. Akbari, Cameron and

*corresponding author, e-mail haemers@uvt.nl

Khosrovshahi [1] give a second construction that transforms a reduced graph of rank r on n vertices (which has to be regular with degree half the number of vertices) into a reduced graph of rank $r+2$ on $2n+2$ vertices. They also give the formulas for the lower bounds when the constructions are applied to K_2 (for the even case) and K_3 (for the odd case).

Proposition 1 $m(r) \geq n(r) = \begin{cases} 2^{(r+2)/2} - 2 & \text{if } r \text{ is even,} \\ 5 \cdot 2^{(r-3)/2} - 2 & \text{if } r \text{ is odd, } r > 1. \end{cases}$

It is conjectured that these inequalities are in fact equalities. Up to rank 8 this has been verified by computer.

2 Constructions

The lower bounds mentioned in Proposition 1 are based upon constructions that transform a reduced graph on n vertices with rank r into a reduced graph on $2n+2$ vertices with rank $r+2$. We present three of such constructions. The first construction is due to Kotlov and Lovász [3] and the second one to Akbari, Cameron and Khosrovshahi [1]. The third one seems to be new.

Construction a ([1, 3]) Let G be a reduced graph on n vertices, adjacency matrix A and rank r. Construct the graph G^a with $2n+2$ vertices and adjacency matrix:

$$\begin{bmatrix} A & A & \mathbf{0} & \mathbf{0} \\ A & A & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & 0 & 1 \\ \mathbf{0} & \mathbf{0} & 1 & 0 \end{bmatrix}$$

($\mathbf{0}$ and $\mathbf{1}$ are the all-zeros and the all-ones row or column vectors, respectively). It is straightforward to check that G^a is reduced and has rank $r+2$.

Construction b ([1]) Let G be a reduced regular graph on n vertices with degree $m = \frac{n}{2}$, adjacency matrix A and rank r. Construct the graph G^b with $2n+2$ vertices and adjacency matrix:

$$\begin{bmatrix} A & A & \mathbf{1} & \mathbf{0} \\ A & A & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & 0 & 1 \\ \mathbf{0} & \mathbf{1} & 1 & 0 \end{bmatrix}.$$

It is straightforward to check that G^b is reduced, regular with degree $n+1$ (half the number of vertices) and has rank $r+2$.

Construction c Let G be a reduced graph on n vertices, adjacency matrix A and rank r. Construct the graph G^c with $2n+2$ vertices and adjacency matrix:

$$\begin{bmatrix} A & \overline{A} & \mathbf{1} & \mathbf{0} \\ \overline{A} & A & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & 0 & 1 \\ \mathbf{0} & \mathbf{1} & 1 & 0 \end{bmatrix},$$

where $\overline{A} = J - A$ (J is the all-ones matrix). It is straightforward to check that G^c is reduced, regular with degree $n+1$ (half the number of vertices) and has rank $r+2$.

Notice that Construction (a) can be applied for any reduced graph and the outcome will never be regular whereas Construction (b) can only be applied (a number of times) to reduced graphs that are regular with degree half the number of vertices. Construction (c) can be applied to any reduced graph and the outcome will always be regular of degree half the number of vertices. In fact, a stronger property holds for the graph G^c.

Definition 2 *We say that a graph G is* $\mathbf{1}$-closed *if it is reduced and for any column of its adjacency matrix there is another column such that the two columns add up to the all-ones vector* $\mathbf{1}$.

Notice the following easy results for a $\mathbf{1}$-closed graph with n vertices.

- For any reduced graph G it holds that G^c is $\mathbf{1}$-closed.
- If G is $\mathbf{1}$-closed then G is regular of degree $\frac{n}{2}$.
- If G is $\mathbf{1}$-closed, then so is G^b.
- If G is $\mathbf{1}$-closed then G^b and G^c are isomorphic.

Although the three constructions are different they sometimes give the same outcome. One example is the case above where Constructions (b) and (c) give the same result if they are applied to a $\mathbf{1}$-closed graph. Another example is the following lemma.

Lemma 1 *Let G be a reduced graph. Then $(G^a)^c$ and $(G^c)^b$ are isomorphic.*

Proof. Let A be the adjacency matrix of G then $(G^a)^c$ has adjacency matrix

$$
\begin{array}{c|cccc|cccc|cc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\cline{2-11}
1 & A & A & \mathbf{0} & \mathbf{0} & \overline{A} & \overline{A} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\
2 & A & A & \mathbf{1} & \mathbf{0} & \overline{A} & \overline{A} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\
3 & \mathbf{0} & \mathbf{1} & 0 & 1 & \mathbf{1} & \mathbf{0} & 1 & 0 & 1 & 0 \\
4 & \mathbf{0} & \mathbf{0} & 1 & 0 & \mathbf{1} & \mathbf{1} & 0 & 1 & 1 & 0 \\
\cline{2-11}
5 & \overline{A} & \overline{A} & \mathbf{1} & \mathbf{1} & A & A & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\
6 & \overline{A} & \overline{A} & \mathbf{0} & \mathbf{1} & A & A & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\
7 & \mathbf{1} & \mathbf{0} & 1 & 0 & \mathbf{0} & \mathbf{1} & 0 & 1 & 0 & 1 \\
8 & \mathbf{1} & \mathbf{1} & 0 & 1 & \mathbf{0} & \mathbf{0} & 1 & 0 & 0 & 1 \\
\cline{2-11}
9 & \mathbf{1} & \mathbf{1} & 1 & 1 & \mathbf{0} & \mathbf{0} & 0 & 0 & 0 & 1 \\
10 & \mathbf{0} & \mathbf{0} & 0 & 0 & \mathbf{1} & \mathbf{1} & 1 & 1 & 1 & 0 \\
\cline{2-11}
\end{array},
$$

If the rows and columns are re-ordered as indicated by the numbers we get the following matrix:

$$
\begin{array}{c|cccc|cccc|cc|}
 & 2 & 5 & 9 & 4 & 1 & 6 & 8 & 10 & 3 & 7 \\
\cline{2-11}
2 & A & \overline{A} & \mathbf{1} & \mathbf{0} & A & \overline{A} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\
5 & \overline{A} & A & \mathbf{0} & \mathbf{1} & \overline{A} & A & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\
9 & \mathbf{1} & \mathbf{0} & 0 & 1 & \mathbf{1} & \mathbf{0} & 0 & 1 & 1 & 0 \\
4 & \mathbf{0} & \mathbf{1} & 1 & 0 & \mathbf{0} & \mathbf{1} & 1 & 0 & 1 & 0 \\
\cline{2-11}
1 & A & \overline{A} & \mathbf{1} & \mathbf{0} & A & \overline{A} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\
6 & \overline{A} & A & \mathbf{0} & \mathbf{1} & \overline{A} & A & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\
8 & \mathbf{1} & \mathbf{0} & 0 & 1 & \mathbf{1} & \mathbf{0} & 0 & 1 & 0 & 1 \\
10 & \mathbf{0} & \mathbf{1} & 1 & 0 & \mathbf{0} & \mathbf{1} & 1 & 0 & 0 & 1 \\
\cline{2-11}
3 & \mathbf{1} & \mathbf{1} & 1 & 1 & \mathbf{0} & \mathbf{0} & 0 & 0 & 0 & 1 \\
7 & \mathbf{0} & \mathbf{0} & 0 & 0 & \mathbf{1} & \mathbf{1} & 1 & 1 & 1 & 0 \\
\cline{2-11}
\end{array}.
$$

which is the adjacency matrix of $(G^c)^b$. $\square$

Corollary 1 *Let G be a reduced graph. Then $(G^a)^c$ and $(G^c)^c$ are isomorphic.*

For the even case, one can start with $G = K_2$ of rank 2 which is $\mathbf{1}$-closed. For each $k \in \{0, 1, 2, \dots, \frac{r-2}{2}\}$ one can first apply k times Construction (c), and then apply $\frac{r-2}{2} - k$ times Construction (a). In this way one finds $\frac{r}{2}$ non-isomorphic reduced graphs of rank r on $n(r) = 2^{(r+2)/2} - 2$ vertices if r is even. For the odd case one can start with $G = K_3$ of rank 3. For each $k \in \{0, 1, 2, \dots, \frac{r-3}{2}\}$ one can first apply k times Construction (c), and then apply $\frac{r-3}{2} - k$ times Construction (a). In this way one finds $\frac{r-1}{2}$ non-isomorphic reduced graphs of rank r on $n(r) = 5 \cdot 2^{(r-3)/2} - 2$ vertices.

Conjecture 1 (Akbari, Cameron and Khosrovshahi [1]) If $r \geq 2$ then $m(r) = n(r)$, and the extremal graphs can be obtained from K_2 or K_3 by Constructions (a), (b) and (c).

We prove that the conjecture is true for two cases. The first case is when r is even and the reduced graph G contains an induced subgraph isomorphic to $\frac{r}{2}K_2$. In this case G has at most $2^{(r+2)/2} - 2$ vertices and if equality holds G is isomorphic to the graph one obtains if Construction (a) is applied $\frac{r}{2} - 1$ times to K_2. The second case is when r is odd and the reduced graph G contains an induced subgraph isomorphic to $K_3 + \frac{r-3}{2}K_2$. In this case G has at most $5 \cdot 2^{(r-3)/2} - 2$ vertices and if equality holds G is isomorphic to the graph one obtains if Construction (a) is applied $\frac{r-3}{2}$ times to K_3.

3 Preliminaries

Let G be a reduced graph with adjacency matrix A of rank r. Then A contains a principle $r \times r$ submatrix A_{11} of rank r which corresponds to an induced subgraph H of G. We call G a *linear extension* of H. We have

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^\top & A_{12}^\top A_{11}^{-1} A_{12} \end{bmatrix} = \begin{bmatrix} A_{11} \\ A_{12}^\top \end{bmatrix} A_{11}^{-1} \begin{bmatrix} A_{11} & A_{12} \end{bmatrix}.$$

This well-known result can be used to design a method to derive all linear extensions of a given graph of full rank r. This method is also described in a slightly different form in [1]:

Step 1 Let H be such a graph, with adjacency matrix B. Then check for all 01-vectors $\boldsymbol{v} \neq \mathbf{0}$ whether $\boldsymbol{v}^\top B^{-1} \boldsymbol{v} = 0$ and keep the vector $\boldsymbol{v}$ if so. Let $(\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k)$ be the list of such vectors. Then this list contains all columns of B. Assume without loss of generality that $B = [\, \boldsymbol{v}_1 \ \boldsymbol{v}_2 \ \cdots \ \boldsymbol{v}_r \,]$.

Step 2 Form an auxiliary graph on the vertex set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$ as follows. For each two distinct indices i and j, put an edge from $\boldsymbol{v}_i$ to $\boldsymbol{v}_j$ if and only if $\boldsymbol{v}_i^\top B^{-1} \boldsymbol{v}_j \in \{0, 1\}$. Notice that any of the vertices $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_r$ is connected to all other vertices. Find all cliques in this graph that contain $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_r$. For a clique C that contains $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_r$, construct a matrix X that contains the vertices of C as columns. Then without loss of generality $X = [\, B \ \ Y \,]$ and

$$A = X^\top B^{-1} X = \begin{bmatrix} B \\ Y^\top \end{bmatrix} B^{-1} \begin{bmatrix} B & Y \end{bmatrix} = \begin{bmatrix} B & Y \\ Y^\top & Y^\top B^{-1} Y \end{bmatrix}$$

is the adjacency matrix of a reduced graph of rank r containing H as a subgraph.

We will use this method in the next section for the special case that r is even and H is the graph $\frac{r}{2}K_2$. In this case we can determine the largest cliques in its auxiliary graph. Secondly we do the same for the special case that r is odd and H is the graph $K_3 + \frac{r-3}{2}K_2$.

4 Main Result

Let m be even. Consider the following $m \times m$ matrix pattern.

$$M_m^* = \begin{bmatrix} 0 & * & * & * & \cdots & \cdots & * & * & * & 1 \\ 0 & 0 & * & * & \cdots & \cdots & * & * & 1 & 0 \\ 0 & 0 & 0 & \ddots & \cdots & \cdots & \iddots & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & * & * & \iddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & 0 & 1 & \iddots & & \vdots & \vdots \\ \vdots & \vdots & & \iddots & 1 & 0 & & & \vdots & \vdots \\ \vdots & \vdots & \iddots & \iddots & \iddots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 1 & 0 & & & & \ddots & \vdots & \vdots \\ 0 & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix}$$

Every column of this matrix stands for a type of 01-vector where two 01-vectors have the same type if they only differ in the $*$-positions. So there is one 01-vector of the type defined by the first column, there are two 01-vectors of the type defined by the second column, four 01-vectors of the type defined by the third column, etcetera. So a column with k $*$-positions defines 2^k 01-vectors. So in total we get

$$2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^{\frac{m}{2}-1} + 2^{\frac{m}{2}-1} + \cdots + 2^3 + 2^2 + 2^1 + 2^0 = 2(2^{\frac{m}{2}} - 1)$$

01-vectors. We define M_r to be the $r \times 2(2^{\frac{r}{2}} - 1)$ matrix whose columns are the vectors defined by the pattern of M_r^*. Let R be the reverse identity matrix of order r (thus R is the adjacency matrix of $\frac{r}{2}K_2$), and define $A_r = M_r^\top R M_r$. It is straightforward to check that A_r is a symmetric 01-matrix with zero's on the diagonal and rank r. So A_r defines a reduced graph of rank r on $n(r) = 2(2^{\frac{r}{2}} - 1)$ vertices. We call this graph G_r.

Theorem 1 *Let r be even, and let G be a linear extension of $\frac{r}{2}K_2$. Then G has at most $n(r)$ vertices. In case of equality G is isomorphic to G_r.*

Proof. Let A be the adjacency matrix of G. Then

$$A = \begin{bmatrix} R & A_{12} \\ A_{12}^\top & A_{12}^\top R^{-1} A_{12} \end{bmatrix} = \begin{bmatrix} R \\ A_{12}^\top \end{bmatrix} R^{-1} \begin{bmatrix} R & A_{12} \end{bmatrix} = Z^\top R Z\,,$$

where $Z = \begin{bmatrix} R & A_{12} \end{bmatrix}$. Now every column $\boldsymbol{z}$ of Z is a 01-vector of length r satisfying $\boldsymbol{z}^\top R \boldsymbol{z} = 0$. Let V be the set of all 01-vectors $\boldsymbol{z} \neq \boldsymbol{0}$ of length r satisfying $\boldsymbol{z}^\top R \boldsymbol{z} = 0$. Then the columns of Z are elements of V. Define a graph $\mathcal{G}$ with vertex set V where two vertices $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are adjacent if $\boldsymbol{z}_1^\top R \boldsymbol{z}_2 \in \{0, 1\}$. Now clearly the columns of Z form a clique in $\mathcal{G}$ and in the previous theorem we proved that $\mathcal{G}$ has a clique of size $n(r) = 2(2^{\frac{r}{2}} - 1)$. We now prove that $\mathcal{G}$ can be colored with $n(r)$ colors proving that there don't exist larger cliques and hence that G has at most $n(r)$ vertices.

Let $\boldsymbol{z} = [z_1, \ldots, z_r]^\top$ be a column of Z, then $\boldsymbol{z}$ is a 01-vector satisfying $\boldsymbol{z}^\top R \boldsymbol{z} = 0$. So if $z_i = 1$ then $z_{r+1-i} = 0$, which means that $z_i + z_{r+1-i} \leq 1$. Hence $\boldsymbol{z}$ has at most $\frac{r}{2}$ coefficients that are equal to 1. Define the following set for $\boldsymbol{z}$:

$$I_{\boldsymbol{z}} = \{i \in \{1, 2, 3, \ldots, \tfrac{r}{2}\} : z_i + z_{r+1-i} = 1\}\,.$$

We call $I_{\boldsymbol{z}}$ the *footprint* of $\boldsymbol{z}$. Now if $I \subseteq \{1, 2, 3, \ldots, \frac{r}{2}\}$ and I has cardinality $k > 0$, then there are 2^k different vectors $\boldsymbol{z} \in V$ with footprint $I_{\boldsymbol{z}} = I$. Furthermore there are $\binom{\frac{r}{2}}{k}$ different subsets of $\{1, 2, 3, \ldots, \frac{r}{2}\}$ of cardinality k. As a consequence

$$|V| = \sum_{k=1}^{\frac{r}{2}} \binom{\frac{r}{2}}{k} 2^k = 3^{\frac{r}{2}} - 1.$$

Out of the 2^k different vectors $\boldsymbol{z} \in V$ with footprint $I_{\boldsymbol{z}} = I$ there are 2^{k-1} vectors for which $|\{i \in \{1, 2, 3, \ldots, \frac{r}{2}\} : z_i = 1\}|$ is even and 2^{k-1} vectors for which $|\{i \in \{1, 2, 3, \ldots, \frac{r}{2}\} : z_i = 1\}|$ is odd. Define

$$\begin{array}{ll} V(I) & = \{\boldsymbol{z} \in V : I_{\boldsymbol{z}} = I\} \\ V(I)_0 & = \{\boldsymbol{z} \in V : I_{\boldsymbol{z}} = I \text{ and } |\{i \in \{1, 2, 3, \ldots, \frac{r}{2}\} : z_i = 1\}| \text{ is even}\} \\ V(I)_1 & = \{\boldsymbol{z} \in V : I_{\boldsymbol{z}} = I \text{ and } |\{i \in \{1, 2, 3, \ldots, \frac{r}{2}\} : z_i = 1\}| \text{ is odd}\} \end{array}$$

Claim: $V(I)_0$ and $V(I)_1$ are cocliques in $\mathcal{G}$. If this claim is proven, the result follows since then the vertex set of $\mathcal{G}$ can be partitioned in the right number of cocliques:

$$\sum_{k=1}^{\frac{r}{2}} \binom{\frac{r}{2}}{k} 2 = 2(2^{\frac{r}{2}} - 1) = n(r).$$

So the clique number of $\mathcal{G}$ is at least $n(r)$ and the chromatic number of $\mathcal{G}$ is at most $n(r)$, so equality must hold in both cases.

In order to prove the claim, let $\boldsymbol{z}_1, \boldsymbol{z}_2 \in V(I)$. So $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ have the same footprint $I_{\boldsymbol{z}_1} = I_{\boldsymbol{z}_2} = I$. First notice that

$$\boldsymbol{z}_1^\top R \boldsymbol{z}_2 = 0 \Longleftrightarrow \boldsymbol{z}_1 = \boldsymbol{z}_2.$$

Now consider

$$(\boldsymbol{z}_1 - \boldsymbol{z}_2)^\top R (\boldsymbol{z}_1 - \boldsymbol{z}_2) = \boldsymbol{z}_1^\top R \boldsymbol{z}_1 + \boldsymbol{z}_2^\top R \boldsymbol{z}_2 - 2\boldsymbol{z}_1^\top R \boldsymbol{z}_2 = -2\boldsymbol{z}_1^\top R \boldsymbol{z}_2.$$

If $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are different at the i-th position, they are also different at the $(r+1-i)$-th position. So if $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are different on p out of the first $\frac{r}{2}$ positions, then $(\boldsymbol{z}_1 - \boldsymbol{z}_2)^\top R (\boldsymbol{z}_1 - \boldsymbol{z}_2) = -2p$. Now p is even if $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are both from $V(I)_0$ or both from $V(I)_1$ and p is odd otherwise. So if $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are both from $V(I)_0$ or both from $V(I)_1$ then $\boldsymbol{z}_1^\top R \boldsymbol{z}_2$ is even. The outcome cannot be zero if $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are different. So $\boldsymbol{z}_1 R \boldsymbol{z}_2 \notin \{0, 1\}$ in this case. This proves the claim.

Finally we have to prove that all cliques of maximum size in $\mathcal{G}$ correspond with a graph that is isomorphic to G_r. For this we make use of the automorphisms of $\frac{r}{2}K_2$. Note that one can permute the $\frac{r}{2}$ edges arbitrarily and that every edge can be flipped. These automorphisms imply transformations on elements of V that do not change their mutual inner products: $\phi(\boldsymbol{z}_1)^\top R \phi(\boldsymbol{z}_2) = \boldsymbol{z}_1^\top R \boldsymbol{z}_2$. We show that each maximum clique in V can be transformed to the maximum clique defined by the pattern of M_r^*.

A clique of maximum size should have one vertex in every coclique of the described coloring. V contains $2^{\frac{r}{2}}$ vectors of weight $\frac{r}{2}$, two of which are in the clique. Since all these vectors correspond to all possible ways to assign 0's and 1's to the vertices of $\frac{r}{2}K_2$ such that every edge has a 1 and a 0, these are all isomorphic. So without loss of generality the two vectors of weight $\frac{r}{2}$ in the maximum clique are the characteristic vectors of $\{1, 2, \ldots, \frac{r}{2}\}$ and (since their inner product should be 0 or 1) $\{1, 2, \ldots, \frac{r}{2} - 1, \frac{r}{2} + 1\}$. Now consider all vectors with footprint $\{1, 2, \ldots, \frac{r}{2} - 1\}$. Two of these are in the clique and since the inner product with the previous two vectors should be 0 or 1, without loss of generality these two vectors are the characteristic vectors of $\{1, 2, \dots, \frac{r}{2} - 1\}$ and $\{1, 2, \dots, \frac{r}{2} - 2, \frac{r}{2} + 2\}$. Now consider all vectors in V with footprint$\{1, 2, \dots, \frac{r}{2} - 2\}$, etcetera.

So without loss of generality (by using the automorphisms of $\frac{r}{2}K_2$), the columns of M_r^*, where every $*$ is replaced by a 1 are part of the maximum clique. All other vertices of the maximum clique are fixed, since there is only one candidate left in each coclique that can be added to this set. Let $I \subseteq \{1, 2, \dots, \frac{r}{2}\}$ and let k be the largest element of I. Consider the vectors with footprint I. Then the only two vectors in V with footprint I that can be in the clique are the characteristic vectors of I and $I \cup \{r - k + 1\} \setminus \{k\}$. The resulting maximum clique is the one defined by the the pattern of M_r^*. $\square$

Theorem 2 *Let $r \geq 5$ be odd, and let G be a linear extension of $K_3 + \frac{r-3}{2}K_2$. Then G has at most $n(r)$ vertices, and there is a unique graph for which equality holds.*

Proof. Let A be the adjacency matrix of G, R the adjacency matrix of $\frac{r-3}{2}K_2$, and let Δ the adjacency matrix of the triangle K_3. Then

$$N = \begin{bmatrix} R & O \\ O & \Delta \end{bmatrix}$$

is the adjacency matrix of $K_3 + \frac{r-3}{2}K_2$, and

$$A = \begin{bmatrix} N & A_{12} \\ A_{12}^\top & A_{12}^\top N^{-1} A_{12} \end{bmatrix} = \begin{bmatrix} N \\ A_{12}^\top \end{bmatrix} N^{-1} \begin{bmatrix} N & A_{12} \end{bmatrix} = Z^\top N^{-1} Z\,,$$

where $Z = \begin{bmatrix} N & A_{12} \end{bmatrix}$. Now every column $\boldsymbol{z}$ of Z is a 01-vector of length r satisfying $\boldsymbol{z}^\top N^{-1} \boldsymbol{z} = 0$. Let V be the set of all 01-vectors $\boldsymbol{z} \neq \boldsymbol{0}$ of length r satisfying $\boldsymbol{z}^\top N^{-1} \boldsymbol{z} = 0$. Then the columns of Z are elements of V. Define a graph $\mathcal{G}$ with vertex set V where two vertices $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are adjacent if $\boldsymbol{z}_1^\top N^{-1} \boldsymbol{z}_2 \in \{0, 1\}$. Now clearly the columns of Z form a clique in $\mathcal{G}$. We now prove that $\mathcal{G}$ can be colored with $n(r)$ colors proving that there don't exist cliques in $\mathcal{G}$ with more than this number of vertices and hence that G has at most $n(r)$ vertices. Let $\boldsymbol{z} = \left[\begin{smallmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{smallmatrix}\right] \in V$ with $\boldsymbol{y} \in \mathbb{R}^3$ and $\boldsymbol{x} \in \mathbb{R}^{r-3}$, then $\boldsymbol{z}$ is a 01-vector satisfying $\boldsymbol{z}^\top N^{-1} \boldsymbol{z} = \boldsymbol{x}^\top R \boldsymbol{x} + \frac{1}{2} \boldsymbol{y}^\top (\Delta - I) \boldsymbol{y} = 0$, which can only happen if $\boldsymbol{x}^\top R \boldsymbol{x} = 0$, and $\boldsymbol{y}^\top (\Delta - I) \boldsymbol{y} = 0$. So

$$\boldsymbol{y} \in \left\{ \boldsymbol{y}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{y}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{y}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\},$$

and hence V has $4 \cdot 3^{\frac{r-3}{2}} - 1$ elements. Let W be the set of all 01-vectors $\boldsymbol{x} \neq \boldsymbol{0}$ of length $r - 3$ satisfying $\boldsymbol{x}^\top R \boldsymbol{x} = 0$. For a vector $\boldsymbol{x} = [x_1, \dots, x_{r-3}]^\top \in W$ we define its footprint $I_{\boldsymbol{x}}$ as

$$I_{\boldsymbol{x}} = \{i \in \{1, 2, 3, \dots, \tfrac{r-3}{2}\} : x_i + x_{r-2-i} = 1\}$$

Now if $I \subseteq \{1, 2, 3, \dots, \frac{r-3}{2}\}$ and I has cardinality $k > 0$ we can define the following sets:

$$\begin{aligned} W(I) &= \{\boldsymbol{x} \in W : I_{\boldsymbol{x}} = I\} \\ W(I)_0 &= \{\boldsymbol{x} \in W : I_{\boldsymbol{x}} = I \text{ and } |\{i \in \{1, 2, 3, \dots, \tfrac{r-3}{2}\} : x_i = 1\}| \text{ is even }\} \\ W(I)_1 &= \{\boldsymbol{x} \in W : I_{\boldsymbol{x}} = I \text{ and } |\{i \in \{1, 2, 3, \dots, \tfrac{r-3}{2}\} : x_i = 1\}| \text{ is odd }\} \end{aligned}$$

Now the following two partitions of V define colorings of $\mathcal{G}$:

Coloring 1. First the three sets

$$\left\{\begin{bmatrix} \mathbf{0} \\ \boldsymbol{y}_1 \end{bmatrix}\right\}, \left\{\begin{bmatrix} \mathbf{0} \\ \boldsymbol{y}_2 \end{bmatrix}\right\}, \left\{\begin{bmatrix} \mathbf{0} \\ \boldsymbol{y}_3 \end{bmatrix}\right\}.$$

Next, define for every $\emptyset \neq I \subseteq \{1, 2, 3, \ldots, \frac{r-3}{2}\}$ the following five sets:

$$\left\{\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_0 \end{bmatrix} : \boldsymbol{x} \in W(I)_0\right\}, \left\{\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_0 \end{bmatrix} : \boldsymbol{x} \in W(I)_1\right\},$$

$$\left\{\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_1 \end{bmatrix} : \boldsymbol{x} \in W(I)_0\right\} \cup \left\{\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_2 \end{bmatrix} : \boldsymbol{x} \in W(I)_1\right\},$$

$$\left\{\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_2 \end{bmatrix} : \boldsymbol{x} \in W(I)_0\right\} \cup \left\{\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_3 \end{bmatrix} : \boldsymbol{x} \in W(I)_1\right\},$$

$$\left\{\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_3 \end{bmatrix} : \boldsymbol{x} \in W(I)_0\right\} \cup \left\{\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_1 \end{bmatrix} : \boldsymbol{x} \in W(I)_1\right\},$$

Coloring 2. First the three sets

$$\left\{\begin{bmatrix} \mathbf{0} \\ \boldsymbol{y}_1 \end{bmatrix}\right\}, \left\{\begin{bmatrix} \mathbf{0} \\ \boldsymbol{y}_2 \end{bmatrix}\right\}, \left\{\begin{bmatrix} \mathbf{0} \\ \boldsymbol{y}_3 \end{bmatrix}\right\}.$$

Next, define for every $\emptyset \neq I \subseteq \{1, 2, 3, \ldots, \frac{r-3}{2}\}$ the following five sets:

$$\left\{\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_0 \end{bmatrix} : \boldsymbol{x} \in W(I)_0\right\}, \left\{\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_0 \end{bmatrix} : \boldsymbol{x} \in W(I)_1\right\},$$

$$\left\{\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_1 \end{bmatrix} : \boldsymbol{x} \in W(I)_0\right\} \cup \left\{\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_3 \end{bmatrix} : \boldsymbol{x} \in W(I)_1\right\},$$

$$\left\{\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_3 \end{bmatrix} : \boldsymbol{x} \in W(I)_0\right\} \cup \left\{\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_2 \end{bmatrix} : \boldsymbol{x} \in W(I)_1\right\},$$

$$\left\{\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_2 \end{bmatrix} : \boldsymbol{x} \in W(I)_0\right\} \cup \left\{\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_1 \end{bmatrix} : \boldsymbol{x} \in W(I)_1\right\}.$$

Similarly as in the previous proof it can be checked that the $5 \cdot (2^{\frac{r-3}{2}} - 1) + 3 = n(r)$ sets of both partitions are cocliques in $\mathcal{G}$.

Now suppose we have a clique in $\mathcal{G}$ of size $n(r)$. Then this clique contains one vertex of each of the cocliques of the colorings. So it contains the three vectors

$$\begin{bmatrix} \mathbf{0} \\ \boldsymbol{y}_1 \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ \boldsymbol{y}_2 \end{bmatrix} \text{ and } \begin{bmatrix} \mathbf{0} \\ \boldsymbol{y}_3 \end{bmatrix}.$$

The clique also contains $2 \cdot (2^{\frac{r-3}{2}} - 1)$ vertices of the form

$$\left\{\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y}_0 \end{bmatrix} : \boldsymbol{x} \in W(I)\right\}.$$

Like in the previous proof it follows that without loss of generality we can take the vertices for which $\boldsymbol{x}$ is a vector that fits the pattern of M^*_{r-3}. Next for every $\emptyset \neq I \subseteq \{1, 2, 3, \ldots, \frac{r-3}{2}\}$ we have three vectors

$$\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{y}_1 \end{bmatrix}, \begin{bmatrix} \boldsymbol{x}_2 \\ \boldsymbol{y}_2 \end{bmatrix} \text{ and } \begin{bmatrix} \boldsymbol{x}_3 \\ \boldsymbol{y}_3 \end{bmatrix}$$

in the clique where either $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in W(I)_0$ or $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in W(I)_1$. It can be verified that without loss of generality $\boldsymbol{x}_1, \boldsymbol{x}_2$ and $\boldsymbol{x}_3$ are all three equal to the unique vector $\boldsymbol{x}$ in $W(I)$ for which $x_i = 0$ for $i > \frac{r-3}{2}$. These cliques of size $n(r)$ all correspond to the graph one gets if one applies Construction (a) $\frac{r-3}{2}$ times to K_3. $\square$

5 Linear extensions

In the previous section we considered maximum linear extensions of $\frac{r}{2}K_2$ and $\frac{r-3}{2}K_2 + K_3$. If Conjecture 1 is correct, then a linear extension of any other graph of full rank r has at most $n(r)$ vertices. For a linear extension of a graph H with adjacency matrix B to be large, we need many vectors $\boldsymbol{v}$ for which $\boldsymbol{v}^\top B^{-1}\boldsymbol{v} = 0$. For most choices of H the number of such vectors is rather small. Especially if B^{-1} contains few zeros, one doesn't expect many such vectors. If H is the complete graph K_r, for example, the only such vectors are the columns of B. Intuitively one expects that the two choices of H considered in this paper have the maximum number of vertices in their maximum linear extensions. Proving this intuitive result, would prove Conjecture 1. Therefore we strongly believe that the conjecture is true.

The following results are straightforward consequences of Constructions (a), (b) and (c).

Proposition 2 *If G is a linear extension of H, then G^a is a linear extension of $H + K_2$.*

As a consequence it follows that the graphs G_r with $n(r)$ vertices, mentioned in Theorem 1 and 2, are isomorphic to the ones obtained by repeatedly applying Construction (a) starting from K_2 (for even r) or K_3 (for odd r). The other constructions are linear extensions of graphs H different from $\frac{r}{2}K_2$ and $\frac{r-3}{2}K_2 + K_3$.

Proposition 3 *Let G be a linear extension of $H = (V, E)$, and let $H' = (V', E')$ with $V' = V \cup \{v_1, v_2\}$ and $E' = E \cup \{\{v, v_1\} : v \in V\} \cup \{v_1, v_2\}$. Then G^a, G^b (if defined) and G^c are linear extensions of H'.*

If G is a reduced graph of rank r, then any induced subgraph of full rank r can be linearly extended to G. In particular, for the unique graph G_r of Theorem 1 every induced subgraph of full rank r has a linear extension with at least $n(r)$ vertices. This leads to the following result.

Theorem 3 *Let r be even and let H be a graph on r vertices having an adjacency matrix B of the following form.*

$$B = \begin{bmatrix} 0 & & & & & & 1 \\ & 0 & & C & & 1 & \\ & & \ddots & & \iddots & & \\ & C^\top & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & & O & \\ & & \iddots & & \ddots & & \\ & 1 & & O & & 0 & \\ 1 & & & & & & 0 \end{bmatrix}.$$

(So, if $H = (V, E)$ and $V = \{1, 2, \ldots, r\}$, this means that if $i + j = r + 1$ then $\{i, j\} \in E$ and if $i + j > r + 1$ then $\{i, j\} \notin E$.) Then H has rank r and G_r is a linear extension of H.

Proof. Clearly rank $B = r$. Let A_r be the adjacency matrix of G_r, and recall that R is the adjacency matrix of $\frac{r}{2}K_2$. Define

$$N = \left[\begin{array}{ccccccc} 0 & & & & & & 1 \\ & 0 & & C & & 1 & \\ & & \ddots & & \iddots & & \\ & O & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & & O & \\ & & \iddots & & \ddots & & \\ & 1 & & O & & 0 & \\ 1 & & & & & & 0 \end{array}\right].$$

Then $B = N^{\top}RN$. Since each column of N is also a column of M_r, we get that H is a subgraph of G_r. $\square$

Thus, because op Property 3 and Theorem 3 we have many graphs of full rank r whose maximum linear extension has at least $n(r)$ vertices. To show that these graphs have at most $n(r)$ vertices could be the most difficult step in the proof of Conjecture 1.

References

[1] S. Akbari, P.J. Cameron and G.B. Khosrovshahi, Ranks and signatures of adjacency matrices, http://designtheory.org/library/preprints/ranks.pdf (preprint).

[2] C.D. Godsil and G.F. Royle, Chromatic Number and the 2-Rank of a graph, *J. Combin. Theory Ser. B* **81** (2001) 142-149.

[3] A. Kotlov and L. Lovász, The rank and size of graphs, *J. Graph Theory* **23** (1996), 185-189.